2009 Vestavia Hills High School Mathematics Tournament Comprehensive Written Examination

1.	Find the area of the triangle formed by the intercepts and the intersection of the asymptotes of the $f(x) = \frac{3}{x-3} - 3$.									
	A. 2	B. 4	C. 8	D. 16	E.	NO	OTA	x		
2.	From all positive integers (x, y) that satisfy $8x + 5y = 107$, find the sum of all the <i>x</i> -values.									
	A. 8	B. 9	C. 13	D. 14	E.	NO	OTA	۱		
3.	Let $A =$ the sum of the elements on the 11 th row of Pascal's triangle, if the 0 th row has one element $B =$ the coefficient of the 4 th term in $(x+7y)^9$. $C =$ the sum of the coefficients of $(4x-2y-3x)^{17}$. D = the number of days in February of 2012 Find $A - \frac{B}{D-1} - C$.									
	A. 3036	B. 1536	C. 1020	D. 0	E.	NO	OTA	۱.		
4.	Given $f(x) = \frac{1}{1+x}$	$\frac{c}{1+ae^{-0.5x}}$, what <i>a</i> -value would give you the same transformation as in $g(x) = \frac{c}{1+e^{-0.5(x-2)}}$								- ?
	A. 1	B. 2	C. <i>e</i>	D. e^2	E. NOTA					
5.	Find $\sum_{n=1}^{10} \begin{vmatrix} 2 & 3 \end{vmatrix}^n$.									
	A. 682	B. 1024	C. 512	D. 876	E. NOTA					
6.	A digit is placed in each empty square in the grid to the right so that each row contains each of the digits $1, 2, 2, 4, 5$ and each column contains each of the									
	digits 1, 2, 3, 4, 5. What digit is placed in the square at the bottom right corner of the grid?					<u> </u>	-			
							5	3		

A. 1 B. 2 C. 3 D. 4 E. 5

7. Luka Bear is a very fuzzy bear that was captured by bear hunters and chained to the corner of the bearhunter outpost as shown. Luka Bear is hungry and must search for bear-berries for food. How much area around the bear-hunter outpost can Luka Bear cover if the chain is 20 feet long? All angles that appear to be right angles are right angles.



A. $225\pi + 50 - 25\pi\sqrt{2}$ B. $335\pi + 50 - 50\pi\sqrt{2}$ C. $365\pi + 50 - 75\pi\sqrt{2}$ D. $400\pi + 50 - 45\pi\sqrt{2}$ E. NOTA

3

1

2

- 8. A plane leaves Vasilieville traveling at 200 mph S 45° E. If a wind blows directly out of the southwest at 100 mph, find the resultant speed in mph.
 - A. $100\sqrt{2}$ B. $100\sqrt{5}$ C. $200\sqrt{5}$ D. $200\sqrt{2}$ E. NOTA
- 9. Find the largest arithmetic mean of x and y if $\prod_{n=1}^{x} (n+5) = \prod_{n=3}^{y+2} (n-2)$, and if x and y are contained on the line that passes through (2005, 2009) and $\left(-\frac{1}{2}, \frac{7}{2}\right)$. A. 116 B. 117 C. 118.5 D. 120.5 E. NOTA
- 10. What is the shortest distance between the point (4, 0) and the parabola $y^2 = 2x$?

A.
$$\frac{\sqrt{2}}{2}$$
 B. $\sqrt{3}$ C. $\sqrt{6}$ D. 4 E. NOTA

- 11. A point *P* is randomly selected from the square with vertices (-2, 2), (2, -2), (-2, -2), and (2, 2). What is the probability that *P* is closer to (0, 0) than it is to (-2, -2)?
- A. $\frac{5}{8}$ B. $\frac{3}{4}$ C. $\frac{7}{8}$ D. $\frac{15}{16}$ E. NOTA 12. Find $X - \frac{\sqrt{3}}{2}$ when $X = \begin{bmatrix} \cos 5\theta & \sin 5\theta \\ -\sin 5\theta & \cos 5\theta \end{bmatrix} \begin{bmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{bmatrix}$ is evaluated at $\theta = \frac{\pi}{48}$. A. $1 - \frac{\sqrt{3}}{2}$ B. $1 + \frac{\sqrt{3}}{2}$ C. 1 D. $\frac{\sqrt{3}}{2} - 1$ E. NOTA
- 13. For how many primes p is $2^{p} + p^{2}$ also prime?
 - A. 0 B. 1 C. 3 D. infinitely many E. NOTA

14. Simplify
$$\frac{1 - \cos^2 x}{\left[\sin^2 x + \tan^2 x + (\sec x)^{-2}\right]^{-1}} + \begin{vmatrix} \cos \frac{\pi}{3} & \sin e \\ \sin 337\pi & -\sec \frac{2}{3}\pi \end{vmatrix}.$$

A. $\sec^2 x$ B. $\csc^2 x$ C. $\tan^2 x$ D. 1 E. NOTA

15. Evaluate $\sum_{k=1}^{100} \frac{\sqrt{306 + \sqrt{306 + \sqrt{306 + ...}}}}{(k+1)(k+2)}$. A. $-\frac{25}{3}$ B. 9 C. $\frac{225}{32}$ D. $\frac{150}{17}$ E. NOTA

- 16. In the figure, line \overrightarrow{AP} bisects $\angle CAB$, line \overrightarrow{BP} bisects $\angle ABC$, $AP = BP = \sqrt{3}$, and CP = 3. What is the area of $\triangle ABC$?
 - A. 4 B. $3\sqrt{3}$ C. $4\sqrt{2}$ D. 6 E. NOTA



- 17. How many different points in three-dimensional space have three positive integral coordinates whose sum is 100?
 - A. 3010 B. 4851 C. 3248 D. 4691 E. NOTA
- 18. Find the area contained by the points of intersection of the graphs of $12x^2 20xy + 7y^2 = 0$ and 2x 3y + 4 = 0. All answers are in square units.

A. 4 B. 8 C. 14 D. 22 E. NOTA

- 19. The following statements were made on the same day:
 - It was Monday yesterday.
 - Today is Thursday.
 - The day after tomorrow will be Friday.
 - Tomorrow will be Saturday.
 - The day before yesterday was Tuesday.

Given that the number of correct statements above uniquely determines the day of the week the statements were made, what day of the week were the statements made?

- A. Monday B. Tuesday C. Wednesday D. Thursday E. NOTA
- 20. In the figure, four semicircles are drawn inside a square with side length 1. Which of the following numbers is closest to the area of the shaded portion?
 - A. 0.3 B. 0.4 C. 0.5 D. 0.6 E. 0.7

21. If $P(x) = 4x^4 + 2x^2 + x$ has roots r_1, r_2, r_3, r_4 , find $\sum_{cyc} \frac{r_1 + r_2}{r_3 + r_4}$.

- A. -6 B. -4 C. $\frac{5}{6}$ D. $\frac{21}{22}$ E. NOTA
- 22. Let A = the sum of the positive factors of 2009 and B = the product of the positive factors of 2009. If $Ax^2 + Bx + C = 0$ has one solution, find C, if A, B, and C are relatively prime and A is positive.
 - A. $\frac{2009}{2394}$ B. $\frac{2009^6}{2}$ C. $\frac{2009^6}{4788}$ D. $\frac{2009^2}{3}$ E. NOTA

23. Find the angle formed by the two lines going through the origin to the points $(4\sqrt{2}, -3\sqrt{3}, 2\sqrt{6})$ and $(-8\sqrt{3}, 9\sqrt{2}, -12)$, where $0 \le \theta < 2\pi$.

- A. π B. $\frac{\pi}{2}$ C. $\frac{3}{4}\pi$ D. $\frac{5}{6}\pi$ E. NOTA
- 24. Dave Stewart of the Eurythmics asked Annie Lennox what the remainder would be when 7⁸⁴ was divided by 5. She incorrectly said 1975 (the year in which the band formed). How far off was she?
 - A. 1971 B. 1972 C. 1973 D. 1974 E. NOTA
- 25. If the minimum number of Friday the 13ths that can occur in a calendar year is V and the maximum number of Friday the 13ths than can occur in a calendar year is H, find V + H.
 - A. 3 B. 5 C. 6 D. 7 E. NOTA



PLEASE WRITE YOUR NAME, COMPLETE SCHOOL NAME, AND TIE-BREAKER ANSWERS ON THE BACK OF THE SCANTRON FORM. DENOTE EACH TIE-BREAKER AS T1, T2, AND T3.

- TB1. If a wheel with radius 8 rotates at 20 revolutions per second, find the angular velocity of the wheel in radians per second.
- TB2. Given that $\cos 40^\circ = a$, compute $\tan 140^\circ$ in terms of a.

TB3. The *x*-value of the solution to $\begin{cases} \frac{4}{x} + \frac{3}{y} = 10\\ \frac{2}{x} - \frac{6}{y} = 6 \end{cases}$ is $\frac{A}{B}$, where *A* and *B* are relatively prime. Find A + B.

YOU MAY KEEP THIS COPY OF THE EXAM.