2008 Hoover High School Mathematics Tournament
Comprehensive Written Test

1. The fraction $\frac{1}{81}$ is written as a decimal of the form $.abc...$ One digit is selected at random from this decimal representation. What is the probability that the digit is a 9?

A) $\frac{1}{3}$ B) $\frac{1}{9}$ C) $\frac{1}{10}$ D) 0 E) NOTA

2. What is the smallest number larger than every term in the sequence $\sqrt{3}, \sqrt{3\sqrt{3}}, \sqrt{3\sqrt{3\sqrt{3}}}, \sqrt{3\sqrt{3\sqrt{3\sqrt{3}}}}, \ldots$

A) 3 B) 2 C) $2\sqrt{2}$ D) $\sqrt{7}$ E) NOTA

3. Mack likes integers with an odd number of positive divisors. Asif likes integers with only odd positive divisors. Salamander likes integers with more than one positive divisor. What is the smallest positive integer all three like?

A) 1 B) 9 C) 25 D) 49 E) NOTA

4. A sphere is inscribed in a cube with volume 192. To the nearest whole percentage, how much of the cube is not occupied by the sphere?

A) 50% B) 52% C) 64% D) 48% E) NOTA

5. A right triangle with hypotenuse of length 4 has one angle of measure 7.5° What is the area enclosed by the triangle?

A) $4\sqrt{2}$ B) $24-6\sqrt{2}-8\sqrt{6}+2\sqrt{3}$ C) $\sqrt{6}-\sqrt{2}$ D) 1 E) NOTA

6. Which of the following prime numbers cannot be expressed as the sum of exactly three positive perfect squares?

A) 149 B) 107 C) 139 D) 173 E) NOTA

7. Ryan has a hover-cow, a cow that can fly. He tethers his hover-cow to an upper outside corner of his rectangular prism house with a 16 foot rope. If Ryan's house is 35 feet deep by 40 feet long by 25 feet tall, how many cubic feet of space can his hover-cow cover?

A) $\frac{14336\pi}{3}$ B) $\frac{8192\pi}{3}$ C) 4096π D) $\frac{11264\pi}{3}$ E) NOTA

8. Simplify: $\sqrt{16777216}$.

A) 4048 B) 4144 C) 4194 D) 4096 E) NOTA

9. Evaluate: $\cos^6 2008 + 3(\cos^4 2008)(\sin^2 2008) + 3(\cos^2 2008)(\sin^4 2008) + \sin^6 2008$.

A) $\cos^2 2008$ B) 1 C) $\sin^2 2008$ D) $1+\sin^2 2008$ E) NOTA

10. Stephanie and Nathan each select a positive number less than 50. Stephanie's number is less than Nathan's number. What is the probability that twice Stephanie's number plus three times Nathan's number is greater than 150?

A) $\frac{2}{3}$ B) $\frac{2}{5}$ C) $\frac{1}{3}$ D) $\frac{4}{25}$ E) NOTA

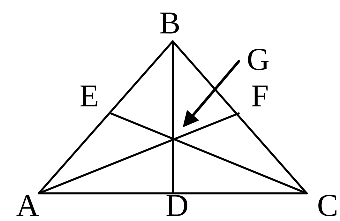
11. Steven and Tyler are standing on a merry-go-round centered at the origin. Steven is standing at the point $(-2,4)$. Tyler dislikes Steven, but not totally, so he stands 210° around the merry-go-round, counterclockwise, from Steven. At what point is Tyler standing?

A) $\left(2+\sqrt{3}, 1-2\sqrt{3}\right)$ B) $\left(\frac{7}{3}, -\frac{10}{3}\right)$ C) $(4,-2)$ D) $\left(0,-2\sqrt{3}\right)$ E) NOTA

12. Let $\vec{L} = <3,2,2>$, $\vec{M} = <3,2,0>$, and $\vec{N} = <0,2,2>$. Find $\vec{N}\cdot\left(\vec{L}\times\vec{M}\right)$.

A) 6 B) -6 C) 12 D) -12 E) NOTA

13. In the figure to the right (not drawn to scale), ABC is a triangle, D lies on side AC, E lies on side AB, F lies on side BC, and G is the intersection of segments AF, CE, and BD. If BF:FC $= 3:4$ and AD:DC $= 7:9$, find EG:EC.



A) 21:13 B) 21:34 C) 21:55 D) 21:76
E) NOTA

14. Solve for w in terms of n: $\sum_{x=1}^{n}\left(\sum_{y=1}^{x}\left(\sum_{z=1}^{y} z\right)\right) = w$

A) $\frac{n^3(n+1)^3}{8}$ B) $\frac{3n^2(n+1)^2}{2}$ C) $\frac{n(n+1)(n+2)(n+3)}{24}$ D) $\frac{n^2(n+1)(n+2)}{6}$ E) NOTA

15. An ellipse is centered at the point $(4,-6)$, has major axis of length 24, and has a distance of 24 from the center to each directrix. What is the area enclosed by the ellipse?

A) 128π B) $84\sqrt{2}\pi$ C) $72\sqrt{3}\pi$ D) $108\sqrt{2}\pi$ E) NOTA

16. Charlie is standing at the point $(0,-1,1)$, Katy is standing at the point $(4,3,2)$, and Jamie is standing at the point $(7,-2,-2)$. They connect their points to form a triangle in order to play E.R.S. Zac is standing at the point $(2,1,0)$. Zac, wants to be included in the game with his friends, but they say they already have a triangle. However, they each will connect with Zac to form a tetrahedron and play in space. Find the volume of this tetrahedron.

A) $6\sqrt{3}$ B) 6 C) $8\sqrt{3}$ D) 8 E) NOTA

17. Ankita asks Alex and Matt to estimate the sum $\sum_{n=0}^{2008}(-1)^n n^2$. Alex guesses 2017016, and Matt guesses 2017096. Ankita tells them that while both of them have the first five digits and the last digit correct, the sixth digit for both is incorrect. What is the correct sixth digit?

A) 2 B) 3 C) 5 D) 7 E) NOTA

18. Let A and B be positive integers. If the functions $f(x) = x^3 - Ax - B$ and $g(x) = x^2 - Ax + B$ have a common integer root, then what is the smallest possible odd value of B?

A) 9 B) 1 C) 17 D) 147 E) NOTA

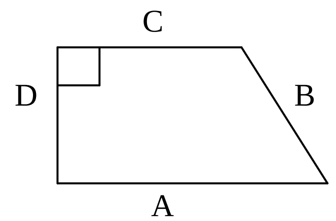
19. Russell is thinking of an arithmetic sequence of integers with common difference 4. Joyce tells Russell that if he adds 5 to the second term, 51 to the fourth term, and some unknown amount to the third term, then the first four terms of the sequence are now in a geometric progression. What must the unknown amount be?

A) 15 B) 13 C) 19 D) 17 E) NOTA

20. A powerful number is defined as any positive integer such that for every prime number p dividing it, p^2 also divides it. How many powerful numbers are in the range $1 \le x \le 1000$?

A) 54 B) 59 C) 31 D) 57 E) NOTA

21. Swaroop, an Überdawg, is tethered to the corner between sides A and B of a building whose trapezoidal base with parallel sides A and C is shown to the right. The tether is $\left(\left(2^3\right)^4\right)^5$ feet long, and side D is 4 feet, side C is 7 feet, and side A is 10 feet. If Swaroop runs around the building counterclockwise until his tether runs out, on which side does Swaroop stop? (Assume the tether always pulls tight around the building, and if he stops at a corner, select NOTA).



A) A B) B C) C D) D E) NOTA

22. If the side length of a regular octagon is $\sqrt{2-\sqrt{2}}$, find the length of the longest line segment that joins two points on the octagon.

A) 1 B) 2 C) $\sqrt{2}$ D) $1+\sqrt{2}$ E) NOTA

23. Find the smallest positive integer x such that $\sqrt{(991)(996)(997)(1002)+x}$ is an integer.

A) 16 B) 784 C) 1024 D) 225 E) NOTA

24. Tausif draws a circle O with center on the positive $x-$axis and tangent to the $y-$axis. He then begins drawing smaller circles inside O with centers also on the positive $x-$axis. The largest of these circles, O_1, has diameter equal to the radius of O and is tangent to O, and for each positive integer i, O_{i+1} has diameter with length equal to the radius of O_i and is tangent to O_i. Also, none of the O_i's are nested (contained inside each other). What fraction of the area contained within O is not contained within any of the O_i's?

A) $\frac{3}{5}$ B) $\frac{7}{10}$ C) $\frac{2}{3}$ D) $\frac{5}{7}$ E) NOTA

25. Find the sum of the series $\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\frac{1}{5}+\frac{1}{6}+\frac{1}{8}+\ldots$, where each term is the reciprocal of an integer whose only prime factors are 2, 3, or 5.

A) $\frac{11}{4}$ B) $\frac{15}{4}$ C) $\frac{41}{15}$ D) $\frac{56}{15}$ E) NOTA

Tiebreakers

TB1. What is the only integer x, with $1<x<100$, that is both a triangular number and a square number?

TB2. Simplify: $\sqrt{2007^2+2006^2-2008^2+4}+3$

TB3. Solve for x: $246.\overline{6}_7 = x_{11}$