Comprehensive Test
VHHS Math Tournament
2007

1. In the complex plane, an ellipse has foci at $1 + 2i$ and $-5 + 2i$. If the ellipse is tangent to the real axis and its area can be expressed as $a\pi\sqrt{b}$, find $a + b$, where b is a square free integer.

A. 7 B. 15 C. 17 D. 9 E. NOTA

2. A polynomial $f(x)$ with integer coefficients has $f(7) = x$ and $f(9) = 13$.
x cannot be:

A. 9 B. 1 C. 3 D. 5 E. NOTA

3. Find the polynomial whose roots are three less than those of $5x^3 + 10x^2 + 2x + 1$.

A. $5x^3 - 35x^2 + 77x - 50$
B. $5x^3 + 97x^2 + 125x + 232$
C. $5x^3 + 55x^2 + 197x + 232$
D. $5x^3 - 30x^2 - 25x + 232$
E. NOTA

4. Find the sum of the reciprocals of the positive divisors of 496.

A. 1 B. 2 C. $\frac{3}{2}$ D. $\frac{5}{2}$ E. NOTA

5. If $f(x)$ is a fifth degree polynomial, and $f(0)=3$, $f(1)=5$, $f(2)=6$, $f(3)=10$, $f(4)=11$, and $f(5)=12$, find $f(6)$.

A. 32 B. 15 C. 14 D. 44 E. NOTA

6. A parallelepiped is determined by vectors: $\langle 5,12,12\rangle$, $\langle 15,9,18\rangle$, $\langle 7,9,16\rangle$. Find the volume of this parallelepiped.

A. 594 B. 864 C. 72 D. 288 E. NOTA

7. Given that x and y are integers and $2x + 5y$ is divisible by 19, which of the following is necessarily divisible by 19?

A. $12x + 4y$ B. $13x + 4y$ C. $12x + 3y$ D. $13x + 3y$ E. NOTA

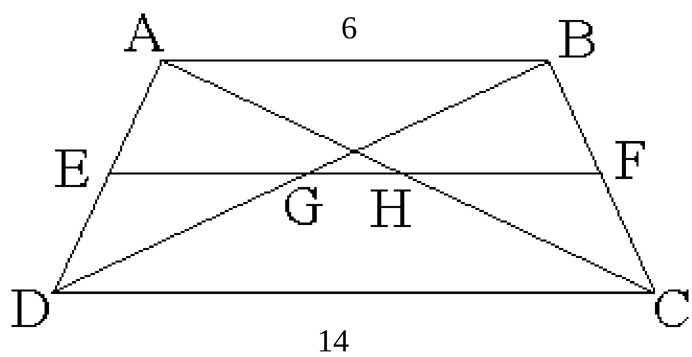
8. A pirate, a ninja, and a zombie play a game where each takes turns drawing out of a hat an integer on the range one to twenty, inclusive. The pirate wins if he draws one of the integers one through eight. The ninja wins if he draws one of the integers nine through thirteen. The zombie wins if he draws fourteen through eighteen. They take turns drawing in the order listed above until someone wins. What is the probability that the zombie wins the game? The integers are replaced after each draw.

A. $\frac{9}{73}$ B. $\frac{9}{100}$ C. $\frac{243}{1000}$ D. $\frac{1143}{5329}$ E. NOTA

9. Write as one trigonometric function in x: $\dfrac{\tan x + \sec x - \cos x}{\sec x + \tan x}$

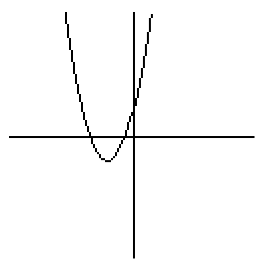
A. $\csc x$ B. $\sin x$ C. $\tan x$ D. $\cot x$ E. NOTA

10. ABCD is a trapezoid. EF is its median. Find the length of segment GH. (Figure is not drawn to scale).

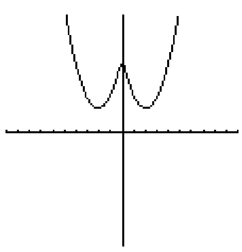
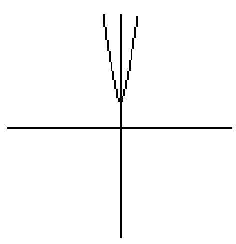
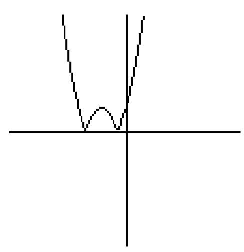
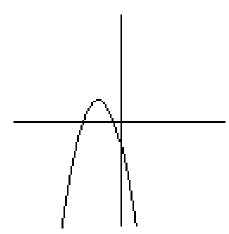


A. 3 B. 2 C. 4 D. 5 E. NOTA

11. Shown below is the graph of $h(x)$. Which of the choices is the graph of $h(|x|)$?



A. B. C. D.

E. NOTA

12. Find the side length of the largest square that will fit completely inside of a 8, 15, 17 right triangle, given that two sides of the square lie on the sides of length 8 and 15?

A. 5 B. $\frac{72\sqrt{5}}{25}$ C. $\frac{120}{23}$ D. $\frac{5}{9}$ E. NOTA

13. Evaluate $\lim_{x\to\infty}\left(\frac{3x}{3x+2}\right)^{x}$.

A. $e^{\frac{3}{2}}$ B. $e^{-\frac{3}{2}}$ C. $e^{-\frac{2}{3}}$ D. $e^{\frac{2}{3}}$ E. NOTA

14. A square can be divided into *n* square sections (not necessarily equal in area), for some positive integers *n*. For example, a 2x2 square can be divided into four 1x1 squares, but such a square cannot be divided into three squares. Thus, a solution exists for *n* = 4. What is the largest *n* for which there is no solution?

A. 11 B. 5 C. 17 D. infinitely large E. NOTA

15. Find the sum $\sum_{i=1}^{99}[0.67i]$ where $[x]$ indicates greatest integer less than or equal to *x*.

A. 3267 B. 3103 C. 4213 D. 3266 E. NOTA

16. An integer between 2 and 99 inclusive is randomly selected, and that number of pennies is placed on a table. Todd and Wes are playing a game in which each takes turns taking 1, 2, 3, 4, 5, 6, 7, 8, or 9 pennies off the table. The person who takes the last penny wins. Assuming each player is intelligent and tries to win, what is the probability that Wes wins if Todd goes first?

A. $\frac{9}{98}$ B. $\frac{1}{2}$ C. $\frac{5}{49}$ D. $\frac{6}{7}$ E. NOTA

17. Find $\cos\left(\frac{2\pi}{7}\right)+\cos\left(\frac{4\pi}{7}\right)+\cos\left(\frac{6\pi}{7}\right)$.

A. $-\frac{1}{4}$ B. $-\frac{1}{2}$ C. $\frac{1}{2}$ D. $\frac{1}{4}$ E. NOTA

18. Eight lasers are positioned at the eight vertices of a convex octagon (not necessarily regular). When a pair of laser beams crosses within the octagon, they form a "void point." What is the most number of void points that can exist in an octagon?

A. 28 B. 57 C. 64 D. 70 E. NOTA

19. Fabio is stuck in Escher's Eerie Amusement Park. At the platform on which he is currently located, he can enter one of six Crazy Slides. Five of the six Crazy Slides loop around and land right back where he started. One of the Crazy Slides takes him out of the park. What is the expected number of slides he must take to get out of the park if Fabio cannot tell which slide he took and chooses a random slide each time?

A. $\frac{11}{5}$ B. $\frac{7}{2}$ C. 6 D. $\frac{49}{6}$ E. NOTA

20. A Kevin Spacey set is any set of numbers S that satisfies the condition that $\left|a_i - a_j\right| > 2$, for all $a_i, a_j \in S$. How many 5 element Kevin Spacey sets are subsets of $\{1, 2, 3, \ldots, 20\}$?

A. 462 B. 512 C. 792 D. 924 E. NOTA

21. How many of the numbers $\binom{200}{k}$, where k is an element of the set $\{0, 1, 2, \ldots, 200\}$ are divisible by 3?

A. 201 B. 36 C. 165 D. 101 E. NOTA

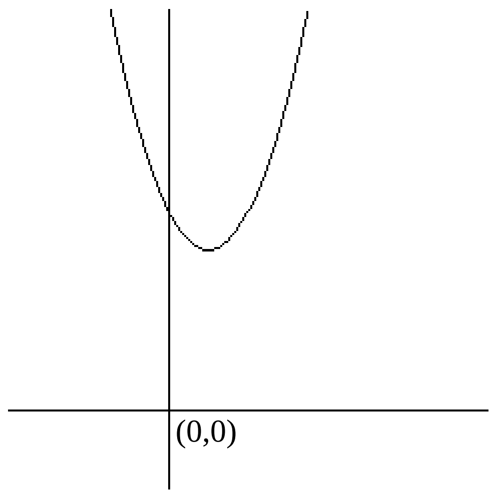
22. In a triangle ABC, with angles A, B, C, sides a, b, c, and $a \ge b \ge c$.
If $\frac{a^3+b^3+c^3}{\sin^3 A+\sin^3 B+\sin^3 C}=8$, find the maximum length of side a.

A. 1 B. $\sqrt{3}$ C. 2 D. $\frac{\sqrt{3}}{2}$ E. NOTA

23. Which one of these is a sixth root of $-i$, that is $\sqrt[6]{-i}$?

A. $-\frac{\sqrt{3}}{2}-\frac{1}{2}i$ B. $\frac{\sqrt{3}}{2}-\frac{1}{2}i$ C. i D. $\frac{\sqrt{3}}{2}+\frac{1}{2}i$ E. NOTA

24. For the parabola $f(x) = ax^2 + bx + c$ shown below, which of the following are true?



I. b is negative
II. c is negative
III. No solutions for $f(x) = 0$ exist in $\mathbb{C}$

A. III only B. I and II only C. I and III only D. I only E. NOTA

25. Compute $\sum_{n=0}^{\infty} \frac{\cos n\theta}{2^n}$ if $\cos\theta = \frac{2}{3}$. (Hint: use the exponential form of $\cos\theta$.)

A. $\frac{6}{7}$ B. $\frac{8}{7}$ C. 1 D. $\frac{4}{5}$ E. NOTA

TB1: In a basketball tournament, each team played exactly one game with every other competitor. Five teams lost two games each and all other teams won two games apiece. There were no ties. How many teams played in the tournament?

TB2: On a particular street, there are n traffic lights, numbered 1 to n. In order to optimize traffic flow, the following condition is applied: With p and q being different numbers, if traffic lights number p and number q have the same color, then traffic light number $p + q$ has a different color. At any given moment, each light is either red, yellow or green. What is the maximum possible value of n?

TB3: If the point (-1, -1) is rotated about the origin with angle $\theta = -\frac{3\pi}{4}$, what will be the location of the point after the rotation?