Hoover High School Mathematics Tournament - March 5, 2005
Comprehensive Written Test

(N.O.T.A. = none of the above)

1. In a math class, four quizzes are given, each counting 15% of the student's final average, as well as a final exam, which counts 40% of the final average. What is a student's final average if she scores 65, 83, 80, and 90 on the quizzes and 92 on the final exam?

   (A) 80.5 (B) 84.5 (C) 86.5 (D) 82 (E) N.O.T.A.

2. Find the last three digits of 23^{2024}.

   (A) 041 (B) 123 (C) 321 (D) 023 (E) N.O.T.A.

3. Which of the following numbers is prime?

   (A) 2001 (B) 2003 (C) 2005 (D) 2007 (E) 2009

4. How far does the tip of the minute hand of a clock move in 35 minutes if the hand is 6 in. long?

   (A) 7 in. (B) $\frac{7\pi}{2}$ in. (C) 14 in. (D) 7π in. (E) N.O.T.A.

5. Find the 210th term of the harmonic sequence $24, 12, 8, 6, \ldots$

   (A) $\frac{1}{24}$ (B) $\frac{5}{24}$ (C) $\frac{4}{35}$ (D) $\frac{1}{14}$ (E) N.O.T.A.

6. How many solutions are there to the equation $4x + 5y = 77$, where x and y are both positive integers?

   (A) 0 (B) 1 (C) 2 (D) 3 (E) N.O.T.A.

7. $(1 - i)^5 =$

   (A) $4 + 4i$ (B) $4 - 4i$ (C) $-4 + 4i$ (D) $-4 - 4i$ (E) N.O.T.A.

8. The population P of a colony of bacteria as a function of time t is given by $P = 20\left(1 + \frac{10}{t}\right)^t$. As time increases, the size of this population approaches what number?

(A) e^{10} (B) $200e$ (C) $20e$ (D) $20e^{10}$ (E) N.O.T.A.

9. California license plates contain exactly 2 letters and exactly 4 digits (including 0), not in any particular order. Letters and digits may be repeated. How many possible license plates are there?

(A) 3,276,000 (B) 6,760,000 (C) 101,400,000 (D) 2,358,720,000
(E) 4,867,200,000

10. The mean and standard deviation of a sample are $\mu = 43$ and $\sigma = 1.1$, respectively. If each of the numbers in this sample is doubled and then increased by 10, find the mean and standard deviation of this new sample.

(A) $\mu = 96$, $\sigma = 12.2$ (B) $\mu = 86$, $\sigma = 12.2$ (C) $\mu = 96$, $\sigma = 2.2$
(D) $\mu = 86$, $\sigma = 2.2$ (E) N.O.T.A.

11. A hyperbola has its center at $(5, 2)$, a vertex at $(5, 5)$, and an asymptote with equation $y = 2x - 8$. Find the distance from the center to one of the foci.

(A) $\frac{3}{2}$ (B) $3\sqrt{3}$ (C) $\frac{3}{2}\sqrt{3}$ (D) $3\sqrt{5}$ (E) $\frac{3}{2}\sqrt{5}$

12. Which of the following is a vector of length 5 in the opposite direction of the vector $< -2, 6, -3 >$?

(A) $< \frac{14}{5}, -\frac{42}{5}, \frac{21}{5} >$ (B) $< 10, -30, 15 >$ (C) $< \frac{10}{7}, -\frac{30}{7}, \frac{15}{7} >$
(D) $< -\frac{10}{49}, \frac{30}{49}, -\frac{15}{49} >$ (E) N.O.T.A.

13. Two cards are dealt from a standard 52-card deck. What is the probability that both cards are the same suit and the same rank?

(A) $\frac{4}{17}$ (B) $\frac{5}{26}$ (C) $\frac{1}{4}$ (D) $\frac{3}{13}$ (E) N.O.T.A.

14. How many positive integers are less than 100 and relatively prime to 100?

(A) 25 (B) 27 (C) 33 (D) 35 (E) 40

15. Which of the following is a sixth root of $-i$?

(A) $\frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2}i$ (B) $-\frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2}i$ (C) $\frac{\sqrt{3}}{2} - \frac{1}{2}i$ (D) $\frac{1}{2} + \frac{\sqrt{3}}{2}i$
(E) N.O.T.A.

16. Suppose that $P(A) = .5$, $P(B) = .3$, and $P(A \cup B) = .6$. What is $P(B|A)$?

(A) .6 (B) .4 (C) .2 (D) .1 (E) N.O.T.A.

17. Let $\alpha = \tan 35°$. Then $\dfrac{\tan 195° - \tan 125°}{1 + \tan 195° \tan 125°} =$

(A) $\frac{2\alpha}{\alpha^2-1}$ (B) $\frac{\alpha}{(1-\alpha)^2}$ (C) $\frac{2\alpha}{1-\alpha^2}$ (D) $\frac{\alpha}{\alpha^2-1}$ (E) N.O.T.A.

18. Find the volume of the solid formed by rotating the triangle with vertices $(-4, 1)$, $(-2, 4)$, and $(-1, 1)$ about the line $y = -3x - 2$.

(A) $\frac{27\pi\sqrt{10}}{10}$ (B) $\frac{729\pi}{80}$ (C) $\frac{81\pi\sqrt{10}}{10}$ (D) $\pi^3 - \tau$ (E) N.O.T.A.

19. Find all real values of x that make $5x - \dfrac{25}{x} + \dfrac{125}{x^3} - \ldots = \dfrac{40}{9}$.

(A) 2 (B) $\frac{8}{9}$ (C) -3 (D) -5 (E) N.O.T.A.

20. Which of the following best describes the graph of $x^2 - 6xy + 9y^2 - 2y - 1 = 0$?

(A) parabola (B) circle (C) ellipse (D) hyperbola
(E) N.O.T.A.

21. Find the area of the triangle with vertices $A(1, -5, 2)$, $B(0, -2, 0)$, and $C(2, -4, 1)$.

(A) $\frac{\sqrt{26}}{2}$ (B) $\sqrt{26}$ (C) $\sqrt{35}$ (D) $2\sqrt{10}$ (E) N.O.T.A.

22. If $A = \begin{bmatrix} 1 & 0 \\ a & 1 \end{bmatrix}$, then $A^{10} =$

(A) $\begin{bmatrix} 1 & 0 \\ a & 1 \end{bmatrix}$ (B) $\begin{bmatrix} 1 & 0 \\ 10a & 1 \end{bmatrix}$ (C) $\begin{bmatrix} 10 & 0 \\ 10a & 10 \end{bmatrix}$ (D) $\begin{bmatrix} 1 & 0 \\ a^{10} & 1 \end{bmatrix}$
(E) N.O.T.A.

23. What common quantity must be added to each term of the sequence $a, 2a^2, 3a^3$ in order to create a geometric sequence?

(A) $-\dfrac{7a^3}{3a^2 + 4a - 1}$ (B) $\dfrac{a^3}{3a^2 - 4a + 1}$ (C) $-\dfrac{a^3}{3a^2 - 4a - 1}$
(D) $\dfrac{7a^3}{3a^2 + 4a + 1}$ (E) N.O.T.A.

24. Which of the following is not a property of binomial coefficients?

(A) $\binom{n}{r} = \binom{n}{n-r}$ (B) $\binom{n}{r} = \frac{n}{r}\binom{n-1}{r-1}$
(C) $\sum_{r=0}^{n} \binom{n}{r} = 2^n$ (D) $\sum_{r=0}^{n} \left[r \binom{n}{r} \right] = n2^{n-1}$
(E) $\sum_{r=0}^{n} \left[r^2 \binom{n}{r} \right] = n^2 2^{n-2}$

25. Find the value of the infinite sum:

$$\frac{1}{6}+\frac{7}{5}+\frac{1}{18}+\frac{7}{72}+\frac{1}{72}+\frac{35}{144}+\frac{1}{324}+\frac{175}{1728}+\ldots$$

(A) $\frac{12}{5}$ (B) $\frac{66}{25}$ (C) $\frac{17}{6}$ (D) $\frac{33}{13}$ (E) N.O.T.A.

Tie Breakers

1. If $f(x)=e^{\ln x}$, then $f(-1)=$

2. Find the eccentricity of the conic section described by $y^2+4y+x+2=0$.

3. If f_k represents the kth Fibonacci number, where $f_1=f_2=1$, and $f_{25}=75025$, find $\sum_{k=1}^{23} f_k$.