Hoover High School Mathematics Tournament - March 5, 2005
Comprehensive Ciphering

1.1 Six students show up late for a movie, leaving only six available adjacent seats (one of which is an aisle seat). If Steve is claustrophobic and must sit on the aisle, and Jackie and Bob must sit next to each other, how many possible seating arrangements are there?

A: 48

1.2 If $1-\sqrt{2}i$ is a solution of the equation $2x^3-7x^2+12x-9=0$, find the equation's other two solutions.

A: $1+\sqrt{2}i,\ \frac{3}{2}$

1.3 The permutation of $\{1,2,3,4\}$ expressed as $\begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 4 & 3 & 1 \end{pmatrix}$ indicates that 1 gets mapped to 2, 2 gets mapped to 4, 3 gets mapped to 3, and 4 gets mapped to 1. How many permutations of $\{1,2,3,4\}$ are there in which no number gets mapped to itself?

A: 9

1.4 Let $A=\begin{bmatrix} 0 & 3 & -1 \\ 3 & -2 & 1 \\ 3 & x & -5 \end{bmatrix}$. If $|A|=0$, find x.

A: 16

1.5 Find the greatest common divisor of 1776 and 1998.

A: 222

2.1 2005_{10} is what number in base 8?

A: 3725 or 3725_8

2.2 Find the value of $\lim\limits_{x\to 1}\dfrac{x^{2005}-1}{x-1}$

A: 2005

2.3 What non-zero real number x makes the vector $< x, 2x, x^2 >$ perpendicular to the vector $< 3, 2, -1 >$?

A: 7

2.4 Find all values of θ, $0 \leq \theta \leq 2\pi$, that satisfy the equation $3\cos^2\theta = \sin^2\theta$

A: $\frac{\pi}{3}, \frac{2\pi}{3}, \frac{4\pi}{3}, \frac{5\pi}{3}$

2.5 A coin is weighted with $P(\text{heads}) = 0.5$, $P(\text{tails}) = 0.4$, and $P(\text{landing on its edge}) = 0.1$. If you flip this coin six times, what is the probability of getting exactly 3 tails?

A: 0.27648 or $\frac{864}{3125}$

3.1 How many positive integral factors does 2491 have?

A: 4

3.2 You have a weighted, six-sided die (numbered 1-6) such that the probability of rolling x is $\frac{x}{21}$. If you roll this die once, what is the expected value of the roll?

A: $\frac{13}{3}$

3.3 A set of five positive integers has a single mode of 10 and a median of 15. What is the smallest possible arithmetic mean of this set?

A: 13.6 or $\frac{68}{5}$

3.4 If $GGGRRRR_{28} = ((813 \cdot 28^4)y + (785 \cdot 29)z)_{10}$, find $y + z$.

A: 43

3.5 What is the minimum value of n such that
$1 + 3 + 9 + ... + 3^{n-1} \geq 20052005$?

A: 16

4.1 Find the conjugate of the reciprocal of the conjugate of $\frac{i}{1+i}$.

A: $1 - i$

4.2 Find the length of the latus rectum of $y^2 - 4x = 6y - 10$.

A: 4

4.3 A pool ball is shot from a corner pocket of a standard 6-pocket, 4 ft. by 8 ft. pool table at an angle of $\tan^{-1}(\frac{3}{10})$ with the longer side. Assuming no friction on the table, how many total bounces off bumpers must occur before the ball lands in a pocket?

A: 6

4.4 What is the cosine of the angle between the space diagonal of a cube and one of the cube's edges?

A: $\dfrac{\sqrt{3}}{3}$

4.5 If $A(x_1, x_2, x_3, x_4, x_5)$ is a lattice point in 5−space with $x_1^2 + x_2^2 + x_3^2 + x_4^2 + x_5^2 \leq 1$, find the maximum possible distance from A to $B(0,1,1,0,1)$?

A: $\sqrt{6}$

Extra 1 A sequence is defined recursively by the formula $a_{n+1} = \dfrac{3a_n}{a_{n-1}}$, where $a_1 = 2$ and $a_2 = 1$. What is the value of a_{2005}?

A: 2

Extra 2 If $x^{x^{x^{x^{\cdots}}}} = 2$, then $(((((x)^x)^x)^x)^x) = ?$

A: 4